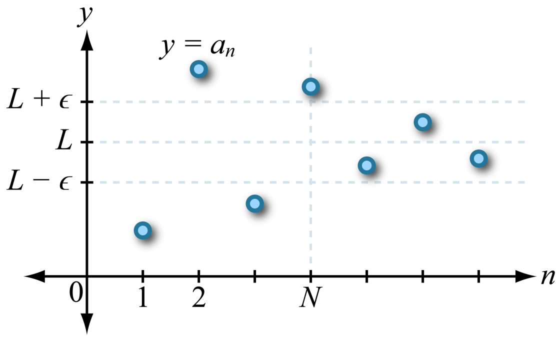
Types of series, convergence tests, and error bounds

Convergence of a sequence



We write

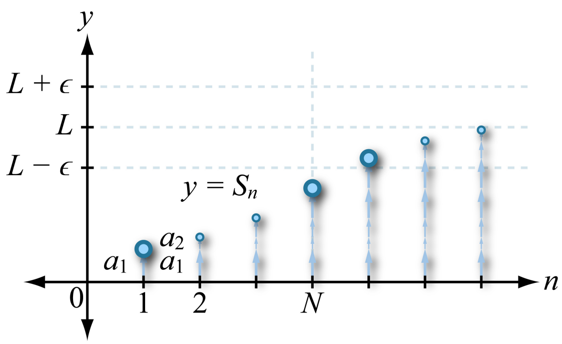
$$\lim_{n\to\infty} a_n = L$$

and say that the sequence (a_n) converges to L

to informally mean that the value a_n can be made to be arbitrarily close to L by requiring n to be a sufficiently large positive number

and to formally mean that so long as ϵ is a positive number, there exists a number N so that trapping integer n in $(N, +\infty)$ guarantees that a_n is trapped in $(L-\epsilon, L+\epsilon)$.

Convergence of a series



We write

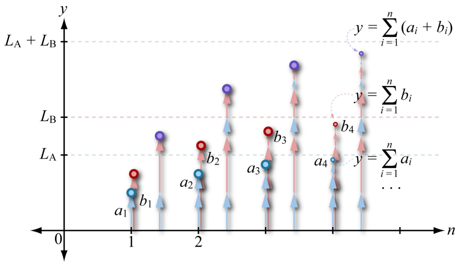
$$\sum_{i=1}^{\infty} a_i := \lim_{n\to\infty} \underbrace{\sum_{i=1}^{n} a_i}_{S_n} = L$$

and say that the series $\sum_{i=1}^{\infty} a_i$ converges to L

to informally mean that the value S_n can be made to be arbitrarily close to L by requiring n to be a sufficiently large positive number

and to formally mean that so long as ϵ is a positive number, there exists a number N so that trapping integer n in $(N, +\infty)$ guarantees that S_n is trapped in $(L-\epsilon, L+\epsilon)$.

Identities



Hypothesis

1. $\sum_{n=1}^{\infty} a_n$ converges
2. $\sum_{n=1}^{\infty} b_n$ converges

Conclusion

Each LHS below converges to the value of the corresponding RHS.

$$\sum_{n=1}^{\infty} c a_n = c \sum_{n=1}^{\infty} a_n , \quad c \text{ a constant}$$

$$\sum_{n=1}^{\infty} (a_n \pm b_n) = \sum_{n=1}^{\infty} a_n \pm \sum_{n=1}^{\infty} b_n$$

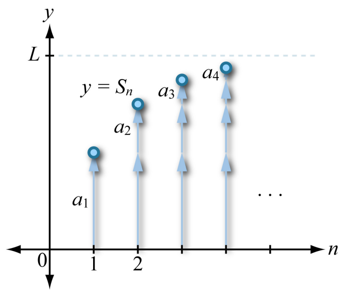
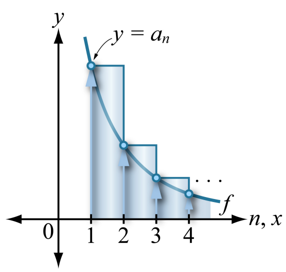
Convergence of a series relates to convergence of its terms

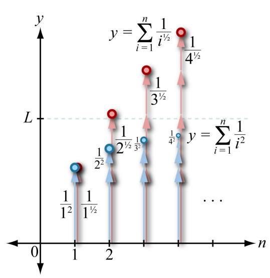
nth term test for divergence		$\lim_{n\to\infty} a_n = 0$	$\lim_{n\to\infty} a_n \neq 0$	$\lim_{n\to\infty} a_n$ DNE
$\sum_{n=1}^{\infty} a_n$	Converges	OK	No	No
	Diverges	OK	OK	OK

Finite number of terms does not affect convergence

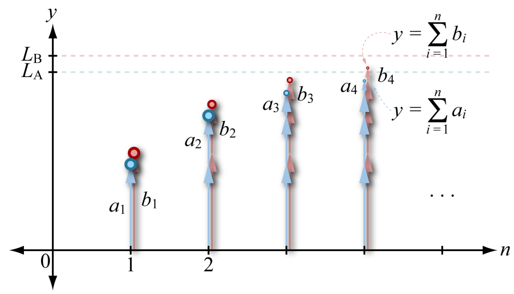
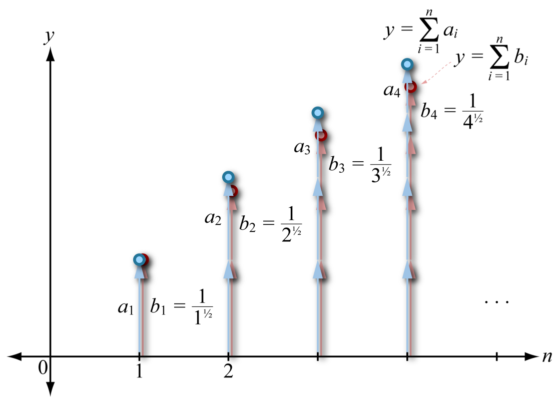
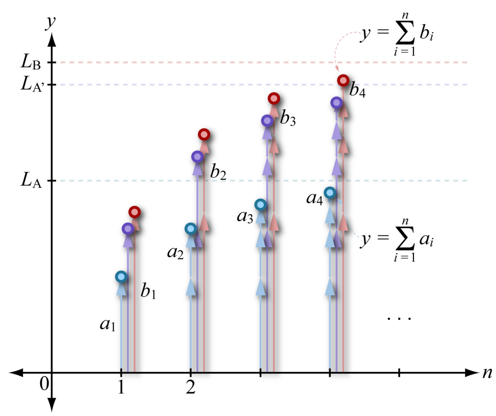
$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{N} a_n + \sum_{n=N+1}^{\infty} a_n , \quad N \in \mathbb{Z}^+$$

Types of series, convergence tests, and error bounds

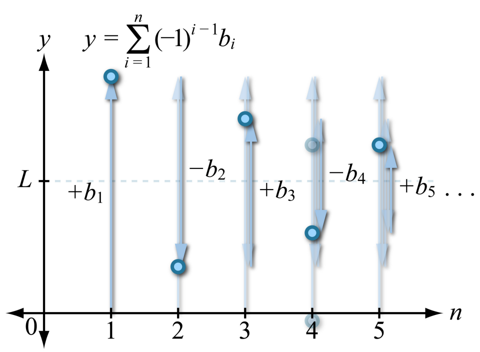
<table>
<tr><th>Descriptions of tests/series</th><th>Conditions</th><th>Conclusion/formula</th></tr>
<tr><td rowspan="2">Geometric
$S = \sum_{n=0}^{\infty} ar^n$

</td><td>$|r| < 1$</td><td>converges
$S = \dfrac{a}{1-r}$</td></tr>
<tr><td>$|r| \geq 1$</td><td>diverges</td></tr>
<tr><td>Integral test
$S = \sum_{n=n_i}^{\infty} a_n$
$a_n = f(n)$
On $[n_i, \infty)$, $f(x)$ is
1. positive
2. decreasing

</td><td colspan="2">$\int_{n_i}^{\infty} f(x)\,\mathrm{d}x$ and $\sum_{n=n_i}^{\infty} a_n$
both converge
- or -
both diverge.

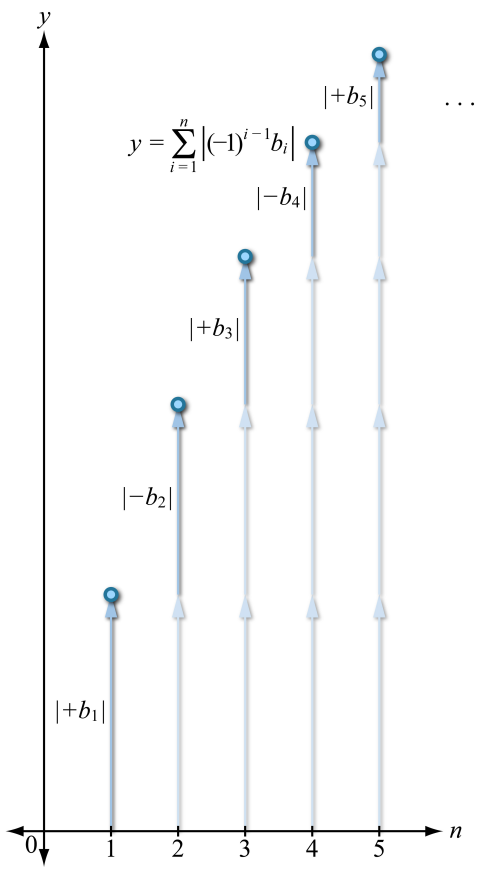
If $\sum_{n=n_i}^{\infty} a_n$ converges by the integral test, then
$$\int_{n+1}^{\infty} f(x)\,\mathrm{d}x \leq \underbrace{S - S_n}_{R_n} \leq \int_{n}^{\infty} f(x)\,\mathrm{d}x$$</td></tr>
<tr><td rowspan="2">p-series
$S = \sum_{n=1}^{\infty} \dfrac{1}{n^p}$
$p > 0$

</td><td>$p > 1$</td><td>converges</td></tr>
<tr><td>$p \leq 1$</td><td>diverges</td></tr>
</table>

Types of series, convergence tests, and error bounds

Descriptions of tests/series		Conditions	Conclusion/formula
Direct comparison test $S_A = \sum_{n=1}^{\infty} a_n$ $S_B = \sum_{n=1}^{\infty} b_n$ Both a_n and $b_n \geq 0$		$\sum_{n=1}^{\infty} b_n$ converges $a_n \leq b_n$	$\sum_{n=1}^{\infty} a_n$ converges
		$\sum_{n=1}^{\infty} b_n$ diverges $a_n \geq b_n$	$\sum_{n=1}^{\infty} a_n$ diverges
Limit comparison test $S_A = \sum_{n=1}^{\infty} a_n$ $S_B = \sum_{n=1}^{\infty} b_n$ Both a_n and $b_n > 0$		$\lim_{n\to\infty} \frac{a_n}{b_n} = c > 0$ where c is a (finite) number	$\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ both converge - or - both diverge
		$\lim_{n\to\infty} \frac{a_n}{b_n} = 0$ and $\sum_{n=1}^{\infty} b_n$ converges	$\sum_{n=1}^{\infty} a_n$ converges
		$\lim_{n\to\infty} \frac{a_n}{b_n} = +\infty$ and $\sum_{n=1}^{\infty} b_n$ diverges	$\sum_{n=1}^{\infty} a_n$ diverges

Types of series, convergence tests, and error bounds





Cancellation between terms can only occur if terms have different signs.

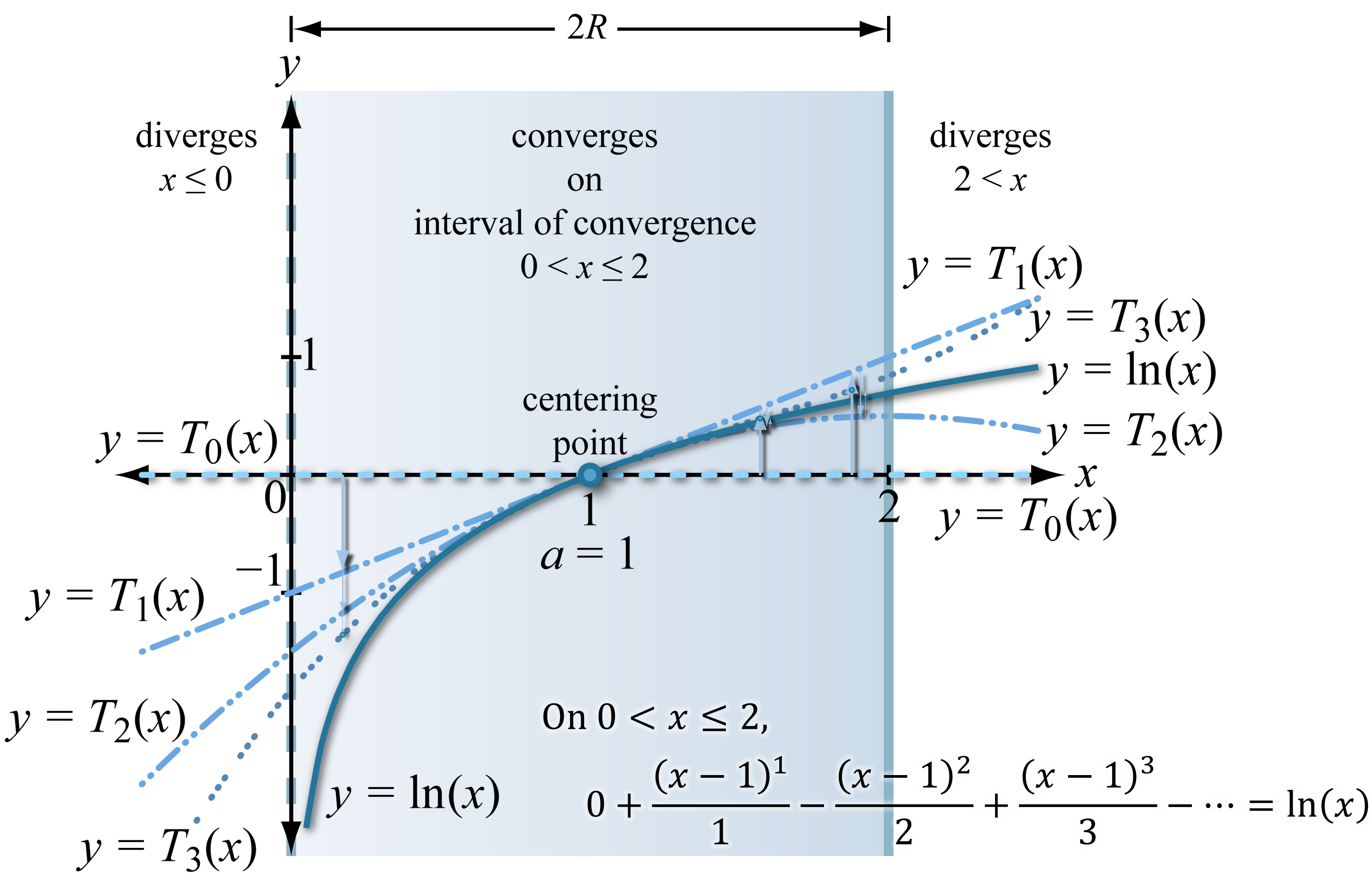
		$\sum_{n=1}^{\infty} \lvert a_n \rvert$ converges	$\sum_{n=1}^{\infty} \lvert a_n \rvert$ diverges
$\sum_{n=1}^{\infty} a_n$	converges	$\sum_{n=1}^{\infty} a_n$ "converges absolutely."	$\sum_{n=1}^{\infty} a_n$ "converges conditionally."
	diverges	Does not happen	$\sum_{n=1}^{\infty} a_n$ diverges.

Descriptions of tests/series	Conditions	Conclusion/formula
Alternating series $S = \sum_{n=1}^{\infty} (-1)^{n-1} b_n$ $S = +b_1 - b_2 + b_3 - b_4 + b_5 - \cdots$ $b_n > 0$	$b_{n+1} \le b_n \forall n$ $\lim_{n\to\infty} b_n = 0$	converges **"Forgotten" error bound** **!!!** usable even in problems where wording inspires use of Lagrange error bound $\lvert R_n \rvert = \lvert S - S_n \rvert \le b_{n+1}$
Ratio test **!!!** often used to find interiors of intervals of convergence of power series $S = \sum_{n=1}^{\infty} a_n$	$\lim_{n\to\infty} \left\lvert \frac{a_{n+1}}{a_n} \right\rvert < 1$	converges absolutely
	$\lim_{n\to\infty} \left\lvert \frac{a_{n+1}}{a_n} \right\rvert > 1$ or $= +\infty$	diverges
Root test $S = \sum_{n=1}^{\infty} a_n$	$\lim_{n\to\infty} \sqrt[n]{\lvert a_n \rvert} < 1$	converges absolutely
	$\lim_{n\to\infty} \sqrt[n]{\lvert a_n \rvert} > 1$ or $= +\infty$	diverges

Power series

$$S = \sum_{n=0}^{\infty} c_n (x-a)^n$$

Possibility	Converges	Iffy	Diverges
1	Only at $x = a$		Everywhere else
2	On $\lvert x-a \rvert < R$ for some $R > 0$	$x - a = \pm R$	On $\lvert x-a \rvert > R$
3	Everywhere		Nowhere



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Taylor series

Series to be molded to imitate function	Function to be imitated	Recipe for imitation
$T(x) = c_0 + c_1(x-a)^1 + c_2(x-a)^2 + c_3(x-a)^3 + c_4(x-a)^4 + \cdots$	$f(x)$	
Features to be molded	Features to be imitated	
$T^{(0)}(a) = c_0 + c_1(a-a)^1 + c_2(a-a)^2 + c_3(a-a)^3 + c_4(a-a)^4 + \cdots$	$f^{(0)}(a)$	
$T'(x) = 0 + c_1 + 2c_2(x-a)^1 + 3c_3(x-a)^2 + 4c_4(x-a)^3 + \cdots$		
$T'(a) = 0 + c_1 + 2c_2(a-a)^1 + 3c_3(a-a)^2 + 4c_4(a-a)^3 + \cdots$	$f'(a)$	
$T''(x) = 0 + 0 + 2 \cdot 1c_2 + 3 \cdot 2c_3(x-a)^1 + 4 \cdot 3c_4(x-a)^2 + \cdots$		
$T''(a) = 0 + 0 + 2 \cdot 1c_2 + 3 \cdot 2c_3(a-a)^1 + 4 \cdot 3c_4(a-a)^2 + \cdots$	$f''(a)$	
$\vdots$	$\vdots$	
$T^{(n)}(a) = n!\,c_n$	$f^{(n)}(a)$	$c_n = \dfrac{f^{(n)}(a)}{n!}$

$$T(x) = f(a) + \frac{f'(a)}{1!}(x-a)^1 + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

Theorems related to convergence of Taylor series to desired function (for college)

Hypothesis

1. $f(x) = T_n(x) + R_n(x)$
2. $T_n(x)$ is the nth-degree Taylor polynomial of f centered at a
3. $\lim_{n\to\infty} R_n(x) = 0$ on $|x-a| < R$

Conclusion

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n = f(x) \text{ on } |x-a| < R$$

Hypothesis

$\exists$ an interval I satisfying the following conditions:

1. Centering value $a \in I$
2. Evaluation value $x \in I$
3. In interval I, f has $n+1$ derivatives

Conclusion

$$\exists z \text{ s.t. } a < z < x \text{ or } x < z < a \text{ s.t. } R_n(x) = \frac{f^{(n+1)}(z)}{(n+1)!}(x-a)^{n+1}$$

Lagrange error bound

!!! Do not have to use this if the alternating series error bound formula will work.

Exploration

Consider a function $f(x)$ having Taylor series $T(x)$ and second-order Taylor polynomial $T_2(x) = f(a) + \frac{f'(a)}{1!}(x-a)^1 + \frac{f''(a)}{2!}(x-a)^2$ centered at $x = a$. Let $f(x)$ and $T_2(x)$ both be evaluated at an $x > a$. How different are $f(x)$ and $T_2(x)$? Study a bound on $f'''(t)$ on an interval $a \le t \le x$ to find out.

$$-\max_{a\le c\le x}|f'''(c)| \le f'''(t) \le \max_{a\le c\le x}|f'''(c)|$$

$$\int_{t=a}^{t=x} -\max_{a\le c\le x}|f'''(c)|\,\mathrm{d}t \le \int_{t=a}^{t=x} f'''(t)\,\mathrm{d}t \le \int_{t=a}^{t=x} \max_{a\le c\le x}|f'''(c)|\,\mathrm{d}t$$

$$-\max_{a\le c\le x}|f'''(c)|\,(x-a) \le f''(x) - f''(a) \le \max_{a\le c\le x}|f'''(c)|\,(x-a)$$

$$\int_{t=a}^{t=x} -\max_{a\le c\le x}|f'''(c)|\,(t-a)\,\mathrm{d}t \le \int_{t=a}^{t=x} f''(t) - f''(a)\,\mathrm{d}t \le \int_{t=a}^{t=x} \max_{a\le c\le x}|f'''(c)|\,(t-a)\,\mathrm{d}t$$

$$\left[-\max_{a\le c\le x}|f'''(c)|\frac{(t-a)^2}{2}\right]_{t=a}^{t=x} \le f'(x) - f'(a) - [f''(a)t]_{t=a}^{t=x} \le \left[\max_{a\le c\le x}|f'''(c)|\frac{(t-a)^2}{2}\right]_{t=a}^{t=x}$$

$$-\max_{a\le c\le x}|f'''(c)|\frac{(x-a)^2}{2} \le f'(x) - f'(a) - f''(a)(x-a) \le \max_{a\le c\le x}|f'''(c)|\frac{(x-a)^2}{2}$$

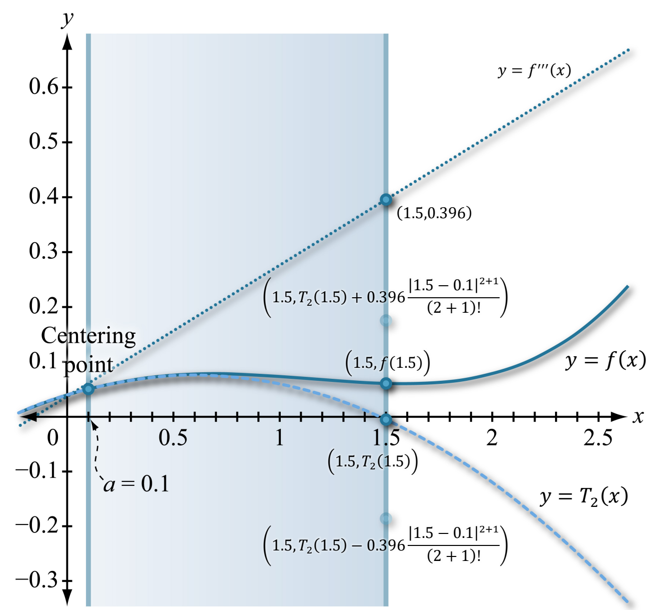
$$\int_{t=a}^{t=x} -\max_{a\le c\le x}|f'''(c)|\frac{(t-a)^2}{2}\,\mathrm{d}t \le \int_{t=a}^{t=x} f'(t) - f'(a) - f''(a)(t-a)\,\mathrm{d}t \le \int_{t=a}^{t=x} \max_{a\le c\le x}|f'''(c)|\frac{(t-a)^2}{2}\,\mathrm{d}t$$

$$\left[-\max_{a\le c\le x}|f'''(c)|\frac{(t-a)^3}{3\cdot 2}\right]_{t=a}^{t=x} \le f(x) - f(a) + \left[-f'(a)t - f''(a)\frac{(t-a)^2}{2}\right]_{t=a}^{t=x} \le \left[\max_{a\le c\le x}|f'''(c)|\frac{(t-a)^3}{3\cdot 2}\right]_{t=a}^{t=x}$$

$$-\max_{a\le c\le x}|f'''(c)|\frac{(x-a)^3}{3!} \le f(x) - f(a) - \frac{f'(a)}{1!}(x-a)^1 - \frac{f''(a)}{2!}(x-a)^2 \le \max_{a\le c\le x}|f'''(c)|\frac{(x-a)^3}{3!}$$

$$-\max_{a\le c\le x}|f^{(2+1)}(c)|\frac{(x-a)^{2+1}}{(2+1)!} \le f(x) - \underbrace{\left[f(a) + \frac{f'(a)}{1!}(x-a)^1 + \frac{f''(a)}{2!}(x-a)^2\right]}_{T_2(x)} \le \max_{a\le c\le x}|f^{(2+1)}(c)|\frac{(x-a)^{2+1}}{(2+1)!}$$

$$-\max_{a\le c\le x}|f^{(2+1)}(c)|\frac{(x-a)^{2+1}}{(2+1)!} \le f(x) - T_2(x) \le \max_{a\le c\le x}|f^{(2+1)}(c)|\frac{(x-a)^{2+1}}{(2+1)!}$$



An analogous pattern involving $T_n(x)$ can be obtained using induction.

$$-\max_{a\le c\le x}|f^{(n+1)}(c)|\frac{(x-a)^{n+1}}{(n+1)!} \le f(x) - T_n(x) \le \max_{a\le c\le x}|f^{(n+1)}(c)|\frac{(x-a)^{n+1}}{(n+1)!}$$

Allowing for the possibility that $x \le c \le a$, we obtain

$$|f(x) - T_n(x)| \le \max_{\substack{a\le c\le x \\ \text{or} \\ x\le c\le a}}|f^{(n+1)}(c)|\frac{|x-a|^{n+1}}{(n+1)!}$$

Taylor's inequality
(as in Stewart, ignore for AP)

!!! Do not have to use this if the alternating series error bound formula will work.

Hypothesis

1. Considering a neighborhood $|x - a| \le d$ containing the centering point.
2. In this neighborhood, the magnitude of the $n+1^{\text{st}}$ derivative of the true function is bounded, e.g. $\left|f^{(n+1)}(x)\right| \le M$, for some (finite) number M.

Conclusion

In this same neighborhood, $|x - a| \le d$, the error of the truncated Taylor polynomial of degree n is also bounded:

$$|R_n(x)| = |f(x) - T_n(x)| \le \frac{M}{(n+1)!}|x - a|^{n+1}$$